

LOWER BOUND FOR THE RANK OF RIGIDITY MATRIX OF 4-VALENT GRAPHS UNDER VARIOUS CONNECTIVITY ASSUMPTIONS

SHISEN LUO

ABSTRACT. In this paper we study the rank of planar rigidity matrix of 4-valent graphs, both in case of generic realizations and configurations in general position, under various connectivity assumptions on the graphs. For each case considered, we prove a lower bound and provide an example which shows the order of the bound we proved is sharp. This work is closed related to work in [L1] and answers some questions raised there.

CONTENTS

1. Introduction	1
2. Proof of Theorem 1.4	7
3. Proof of Theorem 1.1 and Theorem 1.2	10
4. Proof of Theorem 1.3	12
References	15

1. Introduction

Let $G = (V, E)$ be a connected 4-valent graph and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ a planar realization (the words realization and configuration will be used interchangeably). The graph G is always assumed to be finite and simple. The rigidity matrix, which we denote by $R(\mathbf{p})$, is a matrix of size $|E| \times 2|V|$. The rank of $R(\mathbf{p})$, which we denote by $r(G(\mathbf{p}))$, can be taken as the definition of the rank of the *infinitesimal rigidity matroid* of the framework $G(\mathbf{p})$. When \mathbf{p} is a *generic realization*, one can show $r(G(\mathbf{p}))$ is independent of \mathbf{p} as long as it is generic, this can be taken as the definition of the rank of the *generic rigidity matroid* of G and we will write $r(G)$ for it. For more details on the definitions and terminologies in rigidity theory, we refer the readers to [GSS].

Date: July 16, 2012.

2010 Mathematics Subject Classification. Primary: 52C25 Secondary: 05C50.

Key words and phrases. generic rigidity, regular graph.

As we pointed out in [L1], the question of finding the lower bound of the rigidity matrix is closely related to a question in symplectic geometry. We will not repeat it here but refer the readers to [L1] and references therein.

The main results in the paper are summarized as the following four theorems.

Theorem 1.1. *Assume $G = (V, E)$ is a connected 4-valent graph, then*

$$r(G) \geq \frac{8}{5}|V| - 1.$$

Theorem 1.2. *Assume $G = (V, E)$ is a connected 4-valent graph and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ is a configuration in general position, i.e., the image under \mathbf{p} of any three points in V do not lie on the same line, then*

$$r(G(\mathbf{p})) \geq \frac{8}{5}|V| - 1.$$

Theorem 1.3. *Assume $G = (V, E)$ is a connected 4-valent graph which is also 4-edge-connected (see Definition 1.5), then*

$$r(G) \geq \frac{7|V| - 7}{4}.$$

Theorem 1.4. *Assume $G = (V, E)$ is a connected 4-valent graph which is also 4-edge-connected and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ is a configuration in general position, then*

$$r(G(\mathbf{p})) \geq \frac{5|V| - 4}{3}.$$

Definition 1.5. A graph $G = (V, E)$ is called k -edge-connected for some $k \in \mathbb{N}$ if G remains connected upon removing any $k - 1$ edges. In particular, a connected graph is 1-edge-connected.

Theorem 1.1 was Theorem 1 in [L1], here we will give an alternative proof of it by first proving Theorem 1.4 and then Theorem 1.2. Theorem 1.2, Theorem 1.4 and Theorem 1.3 provide answers to Question 1.9 and Question 1.10 in [L1] in case of 4-valent graphs. It should also be pointed out that Theorem 1.4 is essentially proved in Section 5 of [L2], it is the “ $d = 1$ ” version of Theorem 5.1 (or Proposition 5.7) there. However, [L2] was written in a language (mathematically) different from here, so despite the repetition, we will present the full details of the proof in this paper.

Example 1.6. The graph G with configuration \mathbf{p} shown in Figure 1 is Example 1.3 in [L1]. The configuration is slightly modified so that it is in general position. It shows the order of the bound given in Theorem 1.1 and Theorem 1.2 are sharp. Let's briefly recall the construction. Take 3 copies of complete graphs on 5 vertices and delete one edge from each, then connect them together to form a loop as shown in the figure. Both $r(G)$ and $r(G(\mathbf{p}))$ are 24 for this graph. This example can be easily

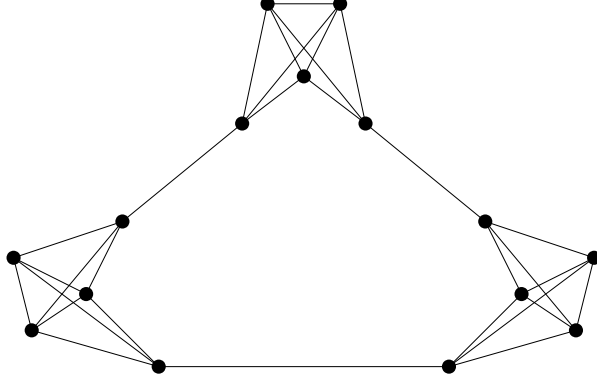


FIGURE 1. An example of 4-valent graph

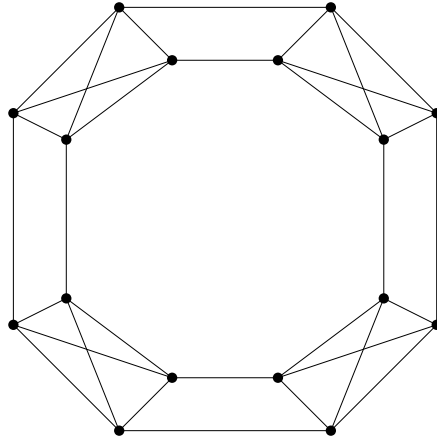


FIGURE 2. An example of 4-valent graph that is 4-edge-connected

generalized to a graph G' of $5k$ vertices and configuration \mathbf{p}' in general position such that both $r(G')$ and $r(G'(\mathbf{p}'))$ are $8k$. In this example, $r(G'(\mathbf{p}'))$ does not depend on \mathbf{p}' as long as \mathbf{p}' is in general position, but it should be emphasized that this is not usually the case.

Example 1.7. The graph G shown in Figure 2 is obtained by connecting 4 copies of complete graphs on 4 vertices. It is 4-edge-connected and of $r(G) = 28$. This example can be easily generalized to a graph G' with $4k$ vertices, such that $r(G') = 7k$ if $k \geq 3$,

$r(G') = 13$ if $k = 2$. This example shows the order of bound given in Theorem 1.3 is sharp.

In order to introduce the example that shows the order of the bound given in Theorem 1.4 is sharp, we first introduce the definition of Cartesian Product in graph theory.

Definition 1.8. the Cartesian product $G \square H$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph such that

- the vertex set of $G_1 \square G_2$ is the Cartesian product $V_1 \times V_2$; and
- any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \square G_2$ if and only if either
 - (a) $u_1 = v_1$ and u_2 is adjacent with v_2 in G_2 , or
 - (b) $u_2 = v_2$ and u_1 is adjacent with v_1 in G_1 .

Lemma 1.9. Assume (G_1, \mathbf{p}_1) and (G_2, \mathbf{p}_2) are two connected graphs with planar configuration in general position. For any non-zero constants $a, b \in \mathbb{R}$, we can define a planar configuration of $G_1 \square G_2$ by

$$\mathbf{p}(u_1, u_2) = a\mathbf{p}_1(u_1) + b\mathbf{p}_2(u_2).$$

If \mathbf{p} is also a configuration in general position, then we have

$$r(G(\mathbf{p})) = r(G_1(\mathbf{p}_1)) + r(G_2(\mathbf{p}_2)) + 2(|V_1| - 1)(|V_2| - 1).$$

Proof. This follows immediately from Proposition 3.18 and Proposition 3.4 in [L2], noting r was denoted by r_1 there. \square

The configuration $\mathbf{p} : V_1 \times V_2 \rightarrow \mathbb{R}^2$ given in the above lemma, which we assume to be in general position, will be denoted by $\mathbf{p}_1 \square \mathbf{p}_2$. The notation is not perfect in that it does not suggest the dependence of \mathbf{p} on the choices of $a, b \in \mathbb{R}$, but this should not bother us since the quantity $r(G(\mathbf{p}))$ does not rely on a, b as the above lemma suggests.

The following example is communicated to me by Prof. Bob Connelly.

Example 1.10. If we use K_l to denote the complete graph on l vertices and P_n to denote a n -gon, i.e., a regular 2-valent graph of n vertices. Then $K_3 \square P_n$ is a 4-edge-connected 4-valent graph. Figure 3(a) and 3(b) give a planar configurations in general position for K_3 and P_6 respectively. Figure 3(c) shows an induced configuration in general position for $K_3 \square P_6$ in the way described in Lemma 1.9. Denote by G the product $K_3 \square P_6$ and by \mathbf{p} the configuration illustrated in Figure 3(c), then according to Lemma 1.9,

$$r(G(\mathbf{p})) = 3 + 6 + 2 \times (3 - 1) \times (6 - 1) = 29.$$

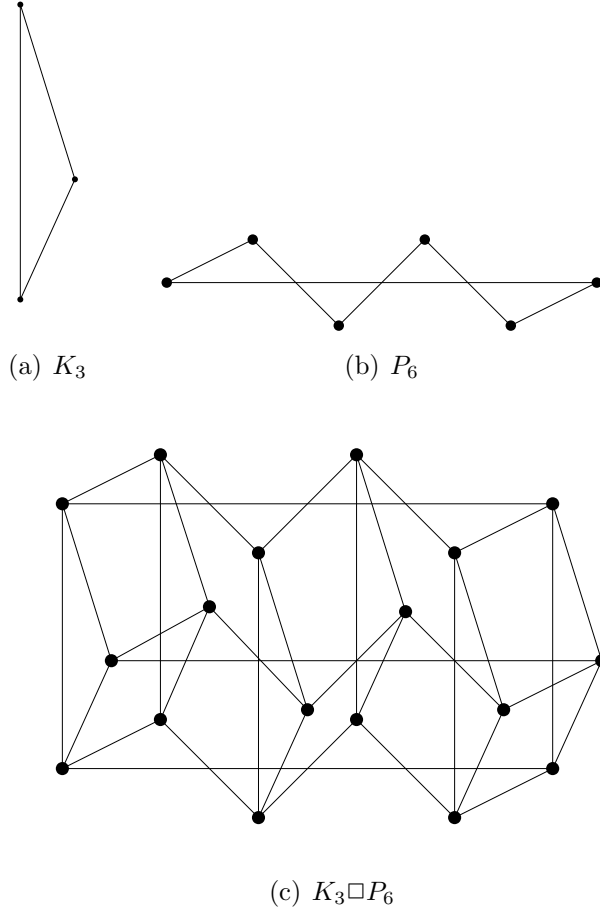


FIGURE 3. A 4-edge-connected 4-valent graph

This example is easily generalized to $G' = K_3 \square P_n$, which is a 4-edge-connected 4-valent graph, equipped with a planar configuration \mathbf{p}' in general position that is induced from a configuration of K_3 and a configuration of P_n , and $r(G'(\mathbf{p}')) = 5k - 1 = \frac{5|V|}{3} - 1$. This class of graphs shows the order of the bound given in Theorem 1.4 is sharp.

It is natural to ask the generalizations of these theorems about regular graphs of degree d for $d \geq 5$. The $d = 5$ version of Theorem 1.1 was proved in [L1]. We ask the following questions.

Question 1.11. Assume $G = (V, E)$ is a connected d -valent graph, $d \geq 6$, do we have

$$r(G) \geq \frac{2d}{d+1}|V| - 1?$$

Note that if the bound holds, then the order of it is sharp. Because we may take $k, k \geq 2$, copies of K_{d+1} , delete one edge from each, then connect them to form a loop as we did in Example 1.6, call the resulting graph G . Then $r(G) = 2dk = \frac{2d}{d+1}m$.

Question 1.12. Assume $G = (V, E)$ is a connected d -valent graph, $d \geq 5$, and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ is a configuration in general position, do we have

$$r(G(\mathbf{p})) \geq \frac{2d}{d+1}|V| - 1?$$

Note that a positive answer to this question would imply the positive answer for Question 1.11 as $r(G) \geq r(G(\mathbf{p}))$. Also, if the bound holds, the order of it is sharp.

Question 1.13. Assume $G = (V, E)$ is a connected d -valent graph which is also d -edge-connected and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ is a configuration in general position, do we have

$$r(G(\mathbf{p})) \geq \frac{(2d-3)m - 2(d-2)}{d-1}?$$

Again, we note that if the bound holds, then the order of it is sharp. We may consider $G = K_{d-1} \square P_n$ with the planar configuration \mathbf{p} in general position induced from that of K_{d-1} and P_n as in Lemma 1.9. Then by Lemma 1.9, we have

$$r(G(\mathbf{p})) = 2(d-1) - 3 + n + 2(d-1-1)(n-1) = 2dn - 3n - 1 = \frac{2d-3}{d-1}m - 1.$$

Also note that the equality holds when $G = K_{d+1}$.

Other than edge-connectivity, there is another kind of connectivity in graph theory called *vertex-connectivity*.

Definition 1.14. A graph $G = (V, E)$ is called k -vertex-connected if for any $U \subseteq V$ with $|U| < k$, the graph $G' = (V', E')$ defined by $V' = V \setminus U$, $E' = E \setminus E_U$ is still connected, where E_U is the subset of E consisting of edges incident to some vertex in U . Note that a connected graph is automatically 1-vertex-connected.

We can ask similar questions about the rank of rigidity matrix with vertex-connectivity assumptions.

Question 1.15. Assume $G = (V, E)$ is a regular d -valent graph that is also k -vertex-connected, $2 \leq k \leq d$. What is the lower bound for $r(G)$? Assume \mathbf{p} is a planar configuration in general position, what is the lower bound for $r(G(\mathbf{p}))$?

In the case of 4-valent graphs, the answer to the above question follows trivially from our results about $r(G)$ and $r(G(\mathbf{p}))$ when assuming (or not) edge-connectivities. We will simply state the result without proof.

Corollary 1.16. *Assume G is a 4-valent graph and \mathbf{p} a planar configuration in general position. If G is 2-vertex-connected, then $r(G) \geq r(G(\mathbf{p})) \geq \frac{8}{5}|V| - 1$. If G is 3-vertex-connected or 4-vertex-connected, then $r(G) \geq \frac{7|V| - 7}{4}$ and $r(G(\mathbf{p})) \geq \frac{5|V| - 4}{3}$. The order of all the above bounds are sharp.*

Acknowledgement: I would like to thank Bob Connelly and Tara Holm for some helpful discussions.

2. Proof of Theorem 1.4

The content in this section is essentially the work in Section 5 of [L2].

First we briefly recall some notations we used and results we obtained in [L1], the readers are referred to Section 2 of [L1] for more details and proofs about these (up to Lemma 2.6 here).

Notation 2.1. Let $G = (V, E)$ be a graph, the vertices are labeled as $v_1, v_2, \dots, v_{|V|}$, the edge incident to v_i and v_j is denoted by e_{ij} . We do not distinguish between e_{ij} and e_{ji} , but most of the time we will make the first coordinate smaller than the second one.

We will use $\lambda(v)$ to denote the degree of v . When the graph G needs to be emphasized (this could be the case if v is the vertex of two graphs under discussion), we use $\lambda_G(v)$ to denote the degree of v as a vertex of G .

We will use E_{v_i} to be subset of E consisting of edges incident to v_i . For any $U \subseteq V$, we will use E_U to denote the set of edges incident to some vertex in U .

The configuration $\mathbf{p} : V \rightarrow \mathbb{R}^2$ will *always* be assumed to be in general position. If G' is a subgraph of G , then by abuse of notation, we will also use \mathbf{p} to denote the configuration of G' induced from that of G .

Definition 2.2. Assume $G = (V, E)$ is a graph and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ is a planar configuration in general position. We use $s(G) = |E| - r(G)$ to denote the *number of stress* of G and use $s_{\mathbf{p}}(G) = |E| - r(G(\mathbf{p}))$ to denote the *number of stress* of $G(\mathbf{p})$.

The following four lemmas and corollary are stated and proved in [L1] under generic configuration assumption, but the proof could be carried over to configuration in general position without any change and we are not going to bother rewriting the proofs.

The following lemma is Lemma 2.3 in [L1] and called the *Deleting Lemma*.

Lemma 2.3. *Given $G = (V, E)$ and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ a planar configuration in general position, assume there is a vertex v_i with $\lambda(v_i) = 1$ or 2 . Define G' to be the graph obtained from G by deleting v_i and E_{v_i} , then $s(G') = s(G)$ and $s_{\mathbf{p}}(G') = s_{\mathbf{p}}(G)$.*

Corollary 2.4. *Given $G = (V, E)$ and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ a planar configuration in general position, let $v_i \in V$. Define G' to be the graph obtained from G by deleting v_i and E_{v_i} , then*

$$0 \leq s(G) - s(G') \leq \max(\lambda(v_i) - 2, 0),$$

and

$$0 \leq s_{\mathbf{p}}(G) - s_{\mathbf{p}}(G') \leq \max(\lambda(v_i) - 2, 0).$$

The following two lemmas are Corollary 2.8 and Corollary 2.9 of [L1].

Lemma 2.5. *Given $G(\mathbf{p})$ a graph with planar configuration in general position, assume deleting one edge e_{ij} increases the number of connected component of G by 1. Denote by G' the graph obtained from G by deleting e_{ij} . Then $s(G') = s(G)$ and $s_{\mathbf{p}}(G') = s_{\mathbf{p}}(G)$.*

Lemma 2.6. *Given $G(\mathbf{p})$ a graph with planar configuration in general position, assume deleting some two edges $e_{i_1j_1}$ and $e_{i_2j_2}$ increases the number of connected components of G by 1, but deleting any one of these two edges does not change the number of connected components. Denote by G' the graph obtained from G by deleting $e_{i_1j_1}$ and $e_{i_2j_2}$. Then $s(G') = s(G)$ and $s_{\mathbf{p}}(G') = s_{\mathbf{p}}(G)$.*

Definition 2.7. A graph $G = (V, E)$ is called *trimmed* if

- each vertex of G is of degree at least 3;
- each connected component of G is 3-edge-connected.

For any graph $G = (V, E)$, we can obtain a trimmed subgraph \tilde{G} of G in the following way. If G is trimmed, then let $\tilde{G} = G$. If G is not trimmed, then we first obtain a subgraph G_1 of G in one of the following two ways:

- (1) If there is vertex v_i of G of degree 1 or 2, let $G_1 = (V \setminus \{v_i\}, E \setminus E_{v_i})$. The choice of v_i may not be unique;
- (2) if every vertex is of degree at least 3, then there exists $F \subseteq E$, such that $|F| \leq 2$, and removing F would increase the number of connected components of G , but removing any proper subset of F would not increase the number of connected components of G , then let $G_1 = (V, E \setminus F)$. Again, the choice of F may not be unique.

If G_1 is trimmed, then we let $\tilde{G} = G_1$. Otherwise, we obtain G_2 as a subgraph of G_1 in the same way as described above. Following this process we get a sequence of graphs $G = G_0, G_1, G_2, \dots, G_q$, where G_{k+1} is a subgraph of G_k obtained in one of two ways we just described, and G_q , which could be empty, is trimmed. We let \tilde{G} be G_q .

Definition 2.8. We call the process of getting \tilde{G} from G described above the *trimming process*. It follows from Lemma 2.3, Lemma 2.5 and Lemma 2.6 that $s(\tilde{G}) = s(G)$ and $s_{\mathbf{p}}(\tilde{G}) = s_{\mathbf{p}}(G)$ for any planar configuration in general position.

Remark 2.9. Even we may have different choices for each graph G_i in the middle of the trimming process, the graph \tilde{G} in fact does not depend on the choice of G_i 's, it is the maximal trimmed subgraph of G . We will not need this, hence will not bother with the proof.

We make the following definition so that the statements and proofs in the rest of this section could be made more concise.

Definition 2.10. We call a graph $G = (V, E)$ is of *type A_4* if

- each vertex of G is of degree 3 or 4;
- each connected component of G has at least one vertex of degree 3;
- each connected component of G is 3-edge-connected.

The following proposition is a special case of Proposition 5.7 of [L2].

Proposition 2.11. Assume $G = (V, E)$ is a graph of type A_4 and $\mathbf{p} : V \rightarrow \mathbb{R}^2$ a planar configuration in general position, then

$$(2.1) \quad s_{\mathbf{p}}(G) \leq \frac{n_4(G)}{3} + c(G),$$

where $n_4(G) = \{v_i \in V \mid \lambda(v_i) = 4\}$ is the number of vertices of degree 4, $c(G)$ is the number of connected components of G .

Proof. We are going to use induction on the size of $|V|$. The graph of type A_4 with least number of vertices is the empty graph, inequality (2.1) holds trivially in this case.

Now we consider $G = (V, E)$ with $|V| = m \geq 1$ and assume (2.1) holds for $|V| < m$. If $c(G) > 1$, then each connected component of G is also of type A_4 and of less vertices. So the induction hypothesis says (2.1) holds for each connected component. We may then add them up to show (2.1) holds for G as well. Now we assume G is connected.

Since G is of type A_4 , there exists a vertex v_t of degree 3. Define $G' = (V', E')$ by $V' = V \setminus \{v_t\}$ and $E' = E \setminus E_{v_t}$. By Corollary 2.4, we have $s_{\mathbf{p}}(G) \leq s_{\mathbf{p}}(G') + 1$.

Apply the trimming process to G' to obtain \tilde{G}' , then $s_{\mathbf{p}}(\tilde{G}') = s_{\mathbf{p}}(G')$ and \tilde{G}' is of type A_4 .

Assume $c(\tilde{G}') = a$, $\tilde{G}' = \bigsqcup_{i=1}^a G_i$ and $G_i = (V_i, E_i)$. Since G is 3-edge-connected, for each G_i , there must be at least 3 vertices v in V_i , such that $\lambda_G(v) = 4$ and $\lambda_{G_i}(v) = 3$.

So

$$\sum_{i=1}^a n_4(G_i) \leq n_4(G) - 3a.$$

Since \tilde{G}' is of type A_4 and of less vertices than G , by the induction hypothesis we have

$$s_{\mathbf{p}}(\tilde{G}') \leq \frac{n_4(\tilde{G}')}{3} + c(\tilde{G}').$$

So

$$\begin{aligned} s_{\mathbf{p}}(G) &\leq s_{\mathbf{p}}(G') + 1 = s_{\mathbf{p}}(\tilde{G}') + 1 \\ &\leq \frac{n_4(\tilde{G}')}{3} + c(\tilde{G}') + 1 \\ &= \frac{1}{3} \sum_{i=1}^a n_4(G_i) + a + 1 \\ &\leq \frac{1}{3} (n_4(G) - 3a) + a + 1 \\ &= \frac{n_4(G)}{3} + 1 = \frac{n_4(G)}{3} + c(G). \end{aligned}$$

This completes the induction step. \square

Proof of Theorem 1.4. Pick one edge $e_{ij} \in E$ and form a new graph $G' = (V, E \setminus \{e_{ij}\})$. By Corollary 2.4 we have $s_{\mathbf{p}}(G) \leq s_{\mathbf{p}}(G') + 1$. The graph G' is of type A_4 , so by Proposition 2.11, we have $s_{\mathbf{p}}(G') \leq \frac{|V| - 2}{3} + 1$, hence $s_{\mathbf{p}}(G) \leq \frac{|V| + 4}{3}$. Therefore

$$r(G(\mathbf{p})) = |E| - s_{\mathbf{p}}(G) \geq 2|V| - \frac{|V| + 4}{3} = \frac{5|V| - 4}{3}.$$

\square

3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.2. If G is the empty graph, the theorem holds trivially. Now we may assume $|V| \geq 5$. If G is 4-edge-connected, then by Theorem 1.4, we have

$$r(G(\mathbf{p})) \geq \frac{5|V| - 4}{3} \geq \frac{8|V|}{5} - 1$$

because $|V| \geq 5$.

If G is not 4-edge-connected, then there exists $F \subseteq E$ with $|F| \leq 3$, such that the graph $G' = (V, E \setminus F)$ is disconnected. We may assume F has the least cardinality among the subsets of E with this property. We observe $|F|$ cannot be 3 or 1, because the sum of degrees of the vertices of one connected component of G' has to be an

even number. So $|F| = 2$. Now apply the trimming process, which was defined in Definition 2.8, to G , we get \tilde{G} . Then \tilde{G} is a proper subgraph of G and is of type A_4 .

We put the connected components of \tilde{G} into two categories: the type of four vertices and the type of at least five vertices. Assume \tilde{G} decomposes into disjoint union of connected components

$$\tilde{G} = \left(\bigsqcup_{i=1}^a G_i \right) \sqcup \left(\bigsqcup_{i=1}^b H_i \right),$$

where each G_i is of four vertices, hence has to be the complete graph on four vertices, each H_i is of at least five vertices and of type A_4 . Assume H_i has m_i vertices, then $n_4(H_i) \leq m_i - 2$, so by Proposition 2.11, we have

$$s_{\mathbf{p}}(H_i) \leq \frac{m_i - 2}{3} + 1.$$

So

$$\begin{aligned} s_{\mathbf{p}}(G) &= s_{\mathbf{p}}(\tilde{G}) = \sum_{i=1}^a s_{\mathbf{p}}(G_i) + \sum_{i=1}^b s_{\mathbf{p}}(H_i) \\ &\leq a + \sum_{i=1}^b \left(\frac{m_i - 2}{3} + 1 \right) \\ &= a + \frac{b}{3} + \frac{1}{3} \sum_{i=1}^b m_i \\ &\leq a + \frac{b}{3} + \frac{1}{3} (|V| - 4a) \\ &= \frac{1}{3} |V| + \frac{b}{3} - \frac{a}{3}, \end{aligned}$$

and because

$$\begin{aligned} &\left(\frac{1}{3} |V| + \frac{b}{3} - \frac{a}{3} \right) - \left(\frac{2}{5} |V| + 1 \right) \\ &= -\frac{1}{15} |V| + \frac{b}{3} - \frac{a}{3} - 1 \\ &\leq -\frac{1}{15} (|V| - 5b) \leq 0 \end{aligned}$$

we immediately see that $s_{\mathbf{p}}(G) \leq \frac{2}{5} |V| + 1$, so

$$r(G(\mathbf{p})) \geq 2|V| - \left(\frac{2}{5} |V| + 1 \right) = \frac{8}{5} |V| - 1.$$

□

Proof of Theorem 1.1. This follows trivially from Theorem 1.2 because we always have $r(G) \geq r(G(\mathbf{p}))$. \square

4. Proof of Theorem 1.3

The proof uses a similar idea with that in the proof of Theorem 1.4. First we recall two results about generic rigidity from Section 2 of [L1].

Lemma 4.1. *Given a graph $G = (V, E)$, assume $F \subseteq E$ with $|F| = 3$ and the edges in F do not share a common vertex. let $G' = (V, E \setminus F)$. If removing F increases the number of connected components of G , but removing any proper subset of F will not do so, then $s(G') = s(G)$.*

Proof. This follows immediately from Lemma 2.7 in [L1]. \square

Lemma 4.2. *Given $G = (V, E)$, assume there is a vertex v_t of degree 3 and the three vertices adjacent to v_t are v_i, v_j and v_k . Assume $e_{ij} \notin E$. We define a new graph $G' = (V', E')$ by $V' = V \setminus \{v_t\}$ and $E' = (E \cup \{e_{ij}\}) \setminus E_{v_t}$. Then $s(G) \leq s(G')$.*

Proof. This is Proposition 2.12 in [L1]. \square

Definition 4.3. A graph $G = (V, E)$ is called *generically trimmed* if

- it is trimmed (see Definition 2.7);
- if there exists $F \subseteq E$, such that $|F| = 3$ and removing F increases the number of connected components of G , then all the edges in F share a common vertex.

Similar to the trimming process, we may define the *generic trimming process*. For any graph $G = (V, E)$, we can obtain a generically trimmed subgraph \tilde{G}^g of G in the following way. If G is generically trimmed, then let $\tilde{G}^g = G$. If G is not generically trimmed, then we first obtain a subgraph G_1 of G in one of the following three ways:

- (1) If there is vertex v_i of G of degree 1 or 2, let $G_1 = (V \setminus \{v_i\}, E \setminus E_{v_i})$. The choice of v_i may not be unique;
- (2) if every vertex is of degree at least 3, and there exists $F \subseteq E$, such that $|F| \leq 2$, and removing F would increase the number of connected components of G , but removing any proper subset of F would not increase the number of connected components of G , then let $G_1 = (V, E \setminus F)$. The choice of F may also not be unique.
- (3) if G is trimmed, and there exists $F \subseteq E$ with $|F| = 3$, such that the vertices in F do not share a common vertex and removing F will increase the number of connected components of G , then let $G_1 = (V, E \setminus F)$. Again, the choice of F may not be unique.

If G_1 is generically trimmed, then we let $\tilde{G}^g = G_1$. Otherwise, we obtain G_2 as a subgraph of G_1 in the same way as described above. Following this process we get a

sequence of graphs $G = G_0, G_1, G_2, \dots, G_q$, where G_{k+1} is a subgraph of G_k obtained in one of three ways we just described, and G_q , which could be empty, is generically trimmed. We let \tilde{G}^g be G_q .

Definition 4.4. We call the process of getting \tilde{G}^g from G described above the *generic trimming process*. It follows from Lemma 2.3, Lemma 2.5, Lemma 2.6 and Lemma 4.1 that $s(\tilde{G}^g) = s(G)$.

Definition 4.5. We call a graph $G = (V, E)$ is of *type B_4* if

- each vertex of G is of degree 3 or 4;
- each connected component of G has at least *four* vertices of degree 3;
- it is generically trimmed.

Lemma 4.6. *Given $G = (V, E)$ is a connected graph of type B_4 , assume $v_t \in V$ is of degree 3 and the three vertices adjacent to v_t are $v_{i_1}, v_{i_2}, v_{i_3}$. Denote by $K_4 = (V', E')$ the complete graph on v_t, v_{i_1}, v_{i_2} and v_{i_3} . If K_4 is a subgraph of G , then $G = K_4$.*

Proof. If $G \neq K_4$, then $F = \{e \in E \setminus E' \mid e \text{ is incident to } v_{i_1}, v_{i_2} \text{ or } v_{i_3}\}$ is not empty. Deleting F from G would disconnect G . Because G is generically trimmed, we must have $|F| = 3$ and there exists $v_x \in V \setminus V'$, such that $\{e_{i_1x}, e_{i_2x}, e_{i_3x}\} \subseteq E$. Because G is of type B_4 , either $\lambda(v_x) = 3$ or $\lambda(v_x) = 4$. If $\lambda(v_x) = 3$, then G only have five vertices: $v_t, v_{i_1}, v_{i_2}, v_{i_3}, v_x$, and only 2 of them are of degree 3, this contradicts the assumption that G is of type B_4 . If $\lambda(v_x) = 4$, assume $v_y \in V \setminus V'$ is the other vertex adjacent to v_x . Then deleting e_{xy} will disconnect G , contradicting the assumption that G is generically trimmed. So $G = K_4$. \square

Proposition 4.7. *If $G = (V, E)$ is of type B_4 , then*

$$(4.1) \quad s(G) \leq \frac{n_4(G)}{4} + c(G),$$

where $n_4(G)$ is the number of vertices of degree 4 in G , $c(G)$ is the number of connected components of G .

Proof. We are going to use induction on $|V|$. The base case is $|V| = 0$, (4.1) holds trivially in this case. Now assume $|V| = m > 0$ and (4.1) holds for graphs with less than m vertices. If $c(G) > 1$, then each connected component of G is of type B_4 and of less vertices, so (4.1) holds for each connected component by the induction hypothesis, we may then add them to show (4.1) holds for G as well. Now we assume G is connected.

Since G is of type B_4 , there exists a vertex v_t of degree 3. Let $G_1 = (V_1, E_1) = (V \setminus \{v_t\}, E \setminus E_{v_t})$. Then by Corollary 2.4, we have $s(G) \leq s(G_1) + 1$.

Case 1: G_1 is of type B_4 . Assume the three vertices adjacent to v_t are $v_{i_1}, v_{i_2}, v_{i_3}$.

- (a) If $\{e_{i_1 i_2}, e_{i_1 i_3}, e_{i_2 i_3}\} \subseteq E$, then by Lemma 4.6 we have $G = K_4$ and we can check (4.1) holds in this case.
- (b) If $\{e_{i_1 i_2}, e_{i_1 i_3}, e_{i_2 i_3}\} \not\subseteq E$, then without loss of generality we may assume $e_{i_1 i_2} \notin E$. Define $G_2 = (V_2, E_2)$ by $V_2 = V \setminus \{v_t\}$ and $E_2 = (E \cup \{e_{i_1 i_2}\}) \setminus E_{v_t}$. Then by Lemma 4.2, we have $s(G) \leq s(G_2)$. Because both G and G_1 are of type B_4 , G_2 must be of type B_4 as well. Since $|V_2| < |V|$, by the induction hypothesis, we have $s(G_2) \leq \frac{n_4(G_2)}{4} + 1$. So $s(G) \leq s(G_2) \leq \frac{n_4(G)}{4} + 1$.

Case 2: G_1 is not of type B_4 .

In this case we apply the generic trimming process to G_1 to obtain \tilde{G}_1^g , then $s(G_1) = s(\tilde{G}_1^g)$. Assume $c(\tilde{G}_1^g) = a$ and $\tilde{G}_1^g = \bigsqcup_{i=1}^a H_i$.

We claim in each H_i , there are at least 4 vertices v such that $\lambda_{H_i}(v) = 3$ and $\lambda_G(v) = 4$. If not, because G is 3-edge-connected, there exist exactly three vertices $v_{j_1}, v_{j_2}, v_{j_3}$ such that $\lambda_{H_i}(v_{j_k}) = 3, \lambda_G(v_{j_k}) = 4$ for $1 \leq k \leq 3$. Assume the edges in G but not in \tilde{G}_1^g and incident to some v_{j_k} are $e_{x j_1}, e_{y j_2}$ and $e_{z j_3}$. Because G is of type B_4 , hence generically trimmed, we must have $x = y = z$. So if $v_x \in V_1$, then v_x is a vertex of \tilde{G}_1^g , but this is not the case, so $v_x \notin V_1$, hence $v_x = v_t$. Then $H_i = G_1$, this contradicts the assumption that G_1 is not of type B_4 . So we have proved the claim.

So each H_i , hence \tilde{G}_1^g , is of type B_4 , by the induction hypothesis, we have

$$s(\tilde{G}_1^g) \leq \frac{n_4(\tilde{G}_1^g)}{4} + c(\tilde{G}_1^g).$$

So

$$\begin{aligned} s(G) &\leq s(G_1) + 1 = s(\tilde{G}_1^g) + 1 \\ &\leq \frac{n_4(\tilde{G}_1^g)}{4} + c(\tilde{G}_1^g) + 1 \\ &\leq \frac{1}{4}(n_4(G) - 4a) + a + 1 \\ &= \frac{n_4(G)}{4} + 1 = \frac{n_4(G)}{4} + c(G). \end{aligned}$$

This completes the induction steps. \square

Proof of Theorem 1.3. Define $G_1 = (V_1, E_1)$ by $V_1 = V \setminus \{v_1\}$ and $E_1 = E \setminus E_{v_1}$. Then by Corollary 2.4, $s(G) \leq s(G_1) + 2$. Apply the generic trimming process to G_1 , we obtain \tilde{G}_1^g , then $s(G_1) = s(\tilde{G}_1^g)$. Assume $c(\tilde{G}_1^g) = a$ and $\tilde{G}_1^g = \bigsqcup_{i=1}^a H_i$. For each H_i , there must exist at least 4 vertices of degree 3, because G is 4-edge-connected.

So each H_i , hence \tilde{G}_1^g , is of type B_4 . By Proposition 4.7, we have

$$s(\tilde{G}_1^g) \leq \frac{n_4(\tilde{G}_1^g)}{4} + a.$$

So

$$\begin{aligned} s(G) &\leq s(G_1) + 2 = s(\tilde{G}_1^g) + 2 \\ &\leq \frac{n_4(\tilde{G}_1^g)}{4} + a + 2 \\ &\leq \frac{1}{4}(|V| - 1 - 4a) + a + 2 \\ &= \frac{|V| + 7}{4} \end{aligned}$$

Hence

$$r(G) = 2|V| - s(G) \geq \frac{7|V| - 7}{4}.$$

□

REFERENCES

- [GSS] J. Graver, B. Servatius and H. Servatius, *Combinatorial rigidity*, **Graduate Studies in Mathematics** Volume 2, **1993**
- [L1] S. Luo, “Lower Bound For The Rank Of 2-Dimensional Generic Rigidity Matroid For Regular Graphs Of Degree Four And Five”, arXiv:1205.4382v2
- [L2] S. Luo, “On Graph Cohomology and Betti Numbers of Hamiltonian GKM manifolds”, arXiv:1206.6111v1

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853-4201, USA
E-mail address: ssluo@math.cornell.edu